# STEP MATHEMATICS 3

# 2022

# Mark Scheme

1. (i) At intersections

$$(ct-a)^2+\left(\frac{c}{t}-b\right)^2=r^2$$

**M1**

Expanding brackets, collecting like terms and multiplying through by $t^2$ $(t \neq 0)$ gives

$$c^2t^4-2ac\,t^3+(a^2+b^2-r^2)\,t^2-2bct+c^2=0$$

as required. ***A1 (2)**

(ii)

$$\sum_{i=1}^{4} {t_i}^2=\left(\sum_{i=1}^{4} t_i\right)^2-2\sum_{i=1}^{3}\sum_{j>i}^{4} t_i\,t_j=\left(\frac{2ac}{c^2}\right)^2-2\frac{(a^2+b^2-r^2)}{c^2}=\frac{2}{c^2}(a^2-b^2+r^2)$$

**M1 A1** **dM1 A1** ***A1 (5)**

as required.

Dividing the equation (*) by $t^4$ (again $t \neq 0$) gives

$$c^2-\frac{2ac}{t}+(a^2+b^2-r^2)-\frac{2bc}{t^3}+\frac{c^2}{t^4}=0$$

which has roots $t_i$ and thus **M1**

$$c^2t^4-2bc\,t^3+(a^2+b^2-r^2)\,t^2-2act+c^2=0$$

**M1 A1**

has roots $\frac{1}{t_i}$, which is just (*) with a and b interchanged. **E1**

Thus

$$\sum_{i=1}^{4}\frac{1}{{t_i}^2}=\frac{2}{c^2}(b^2-a^2+r^2)$$

from the first result of (ii). **A1 (5)**

**Alternative:-**

$$\sum_{i=1}^{4}\frac{1}{{t_i}^2}=\frac{{t_1}^2{t_2}^2{t_3}^2+{t_2}^2{t_3}^2{t_4}^2+{t_3}^2{t_4}^2{t_1}^2+{t_4}^2{t_1}^2{t_2}^2}{{t_1}^2{t_2}^2{t_3}^2{t_4}^2}$$

**M1**

$$=\frac{(t_1t_2t_3+t_2t_3t_4+t_3t_4t_1+t_4t_1t_2)^2-2t_1t_2t_3t_4(t_1t_2+t_1t_3+t_1t_4+t_2t_3+t_2t_4+t_3t_4)}{(t_1t_2t_3t_4)^2}$$

**M1 A1 A1**

$$=\frac{2}{c^2}(b^2-a^2+r^2)$$

**A1**

(iii)

$$\sum_{i=1}^{4} OP_i{}^2 = \sum_{i=1}^{4}\left(c^2 t_i{}^2 + \frac{c^2}{t_i{}^2}\right) = c^2\left(\frac{2}{c^2}(a^2 - b^2 + r^2) + \frac{2}{c^2}(b^2 - a^2 + r^2)\right) = 4r^2$$

as required. **M1 *A1 (2)**

(iv) Touching at two distinct points implies the roots of (*) are two pairs of coincident roots.

WLOG say $t_1 = t_3$ and $t_2 = t_4$ . **E1**

then as the product of the four roots is 1 (from (*)), $t_1{}^2 t_2{}^2 = 1$ **B1** and therefore $t_1 t_2 = \pm 1$ .

$P_1$ is $\left(ct_1, \frac{c}{t_1}\right)$ and $P_2$ is $\left(ct_2, \frac{c}{t_2}\right) = \pm\left(\frac{c}{t_1}, ct_1\right)$ **B1** which are reflections of one another in $y = \pm x$ respectively, and these are the mediators of the pairs of points. **E1** The centre of the circle $C_2$ lies on the mediator of $P_1$ and $P_2$ **E1** which we have shown is $y = \pm x$ . **E1 (6)**

**Alternative E1 B1 as before**

$$t_1 + t_2 + t_3 + t_4 = \frac{2a}{c} \Rightarrow t_1 + t_2 = \frac{a}{c}$$

$$t_1t_2t_3 + t_2t_3t_4 + t_3t_4t_1 + t_4t_1t_2 = \frac{2b}{c} \Rightarrow t_1{}^2t_2 + t_2{}^2t_1 + t_1{}^2t_2 + t_2{}^2t_1 = \frac{2b}{c}$$

$$\Rightarrow (t_1 + t_2)t_1t_2 = \frac{b}{c}$$

**M1 A1**

So $t_1t_2 = \frac{b}{a} = \pm 1$ and hence $a = \pm b$ and so the centre of $C_2$ is $(a, \pm a)$ as required.

**M1 A1**

**Alternative E1 B1 as before**

$$t_1{}^2 + t_2{}^2 + t_3{}^2 + t_4{}^2 = 2(t_1{}^2 + t_2{}^2) = 2\left(\frac{1}{t_2{}^2} + \frac{1}{t_1{}^2}\right) = \frac{1}{t_1{}^2} + \frac{1}{t_2{}^2} + \frac{1}{t_3{}^2} + \frac{1}{t_4{}^2}$$

**M1 A1**

and thus $\frac{2}{c^2}(a^2 - b^2 + r^2) = \frac{2}{c^2}(b^2 - a^2 + r^2)$ so $a^2 = b^2$ and $a = \pm b$

**M1 A1**

2. (i)

If

$$a^3 + 2b^3 + 4c^3 = 0$$

then

$$a^3 = 0 - 2b^3 - 4c^3 = 2(-b^3 - 4c^3)$$

which is even. If $a$ were odd, then $a^3$ would be odd. So, $a$ is even.

Thus $\exists p$ where $2p = a$, with $p$ an integer and $|p| < |a|$ **E1**

Substituting for $a$ in the original equation, $8p^3 + 2b^3 + 4c^3 = 0$ . Dividing by 2 and rearranging gives $b^3 + 2c^3 + 4p^3 = 0$ which is the original equation with a,b,c replaced by b,c p.

So we may repeat the argument with, say, $2q = b$ and then having done so repeat the whole argument with $2r = c$ . **E1**

Thus $\exists p, q, r$ integers with $2r = c$ , $|r| < |c|$ and $p^3 + 2q^3 + 4r^3 = 0$ .

So if there were to be a set of such integers a, b, c, there would be a set of such integers p, q, r with smaller modulus satisfying the same result. This argument may be repeated ad infinitum leading to the conclusion that there is no least modulus set of integers which is not possible as an infinitely decreasing sequence of positive integers cannot exist being bounded by 1. (alternatively, assume a, b, c to be smallest modulus, then we have a contradiction) **E1** Hence no such a, b, c exist. ***B1 (4)**

(ii) If $9a^3 + 10b^3 + 6c^3 = 0$ , then $10b^3 = -9a^3 - 6c^3 = 3(-3a^3 - 2c^3)$

Thus $10b^3$ is a multiple of 3 and so, $b^3$ is a multiple of 3 and thus, $b$ is a multiple of 3.

Thus $\exists q$ where $3q = b$, with $q$ an integer and $|q| < |b|$ **M1 A1** and $9a^3 + 270q^3 + 6c^3 = 0$ which can be divided by 3 to give $3a^3 + 90q^3 + 2c^3 = 0$.

It would follow that $2c^3 = -3a^3 - 90q^3 = 3(-a^3 - 30q^3)$ and so $\exists r$ where $3r = c$, with $r$ an integer and $|r| < |c|$.

Substituting for $c$ , $3a^3 + 90q^3 + 54r^3 = 0$ leading to $a^3 + 30q^3 + 18r^3 = 0$ .

We may repeat the argument with $3p = a$ leading to $27p^3 + 30q^3 + 18r^3 = 0$ which on division by 3 gives $9p^3 + 10q^3 + 6r^3 = 0$ , the original equation with a, b, c replaced by p, q, r. **A1**

So the conclusion can be drawn in the same way as in part (i). ('by descent') **E1 (4)**

(iii) $(3n \pm 1)^2 = 9n^2 \pm 6n + 1 = 3(3n^2 \pm 2n) + 1$ **B1** Every integer may be written as $3n - 1$ , $3n$ or $3n + 1$ . We have shown that the square of an integer which is not a multiple of 3 is one more than a multiple of 3, and if an integer is a multiple of 3 then it can be written $3n$ and

$(3n)^2 = 9n^2 = 3(3n^2)$ which is a multiple of 3. Thus the sum of two integers can only be either a multiple of 3, one more than a multiple of 3, or two more than a multiple of 3 depending on whether the two integers are multiples of 3, exactly one is a multiple of 3 or neither is a multiple of 3 respectively. Hence the result that the sum of two squares can only be a multiple of three if each of the integers is a multiple of three. **E1**

If $a^2 + b^2 = 3c^2$ , by the result just deduced,

$\exists p, q$ where $3p = a$ , $3q = b$ and $|p| < |a|$ , $|q| < |b|$ **M1**

Substituting for a and b, $9p^2 + 9q^2 = 3c^2$ so $c^2 = 3(3p^2 + 3q^2)$ meaning that $c^2$ is a multiple of 3 and hence $c$ is a multiple of 3.

So $\exists r$ where $3r = c$, with $r$ an integer and $|r| < |c|$ , and substituting for c and dividing by 9 ,

$p^2 + q^2 = 3r^2$ which is the original with a, b ,c replaced by p, q, r. As in (i) and (ii), the required result follows by descent. **E1 (4)**

(iv) $(4n \pm 1)^2 = 16n^2 \pm 8n + 1 = 4(4n^2 \pm 2n) + 1$ so, the square of an odd integer is one more than a multiple of four. **M1** $(2n)^2 = 4n^2$ so the square of an even integer is a multiple of four. **M1**

Thus, the sum of the squares of three non-zero integers must be 0, 1, 2 or 3 more than a multiple of four as the integers are all even, two even and one odd, one even and two odd, or all odd respectively. **A1**

Thus if $a^2 + b^2 + c^2 = 4abc$ , $a, b$ , and $c$ must all be even. **B1**

Thus $\exists p, q, r$ integers with $2p = a$ , $2q = b$ , $2r = c$ , and $|p| < |a|$ , $|q| < |b|$ , $|r| < |c|$ . **M1**

So, if $a^2 + b^2 + c^2 = 4abc$ , $4p^2 + 4q^2 + 4r^2 = 32pqr$ , which simplifies to

$p^2 + q^2 + r^2 = 8pqr$. (Alternatively, $a^2 + b^2 + c^2 = 2^n abc$ , $a^2 + b^2 + c^2 = 2^{n+1} abc$ )

**M1**

The argument can be repeated with $p, q$ , and $r$ all being even integers with the multiple of the RHS being a power of two greater than 4. **E1** Thus the result follows by descent. **E1 (8)**

3. (i) $ax^2 + bxy + cy^2 = 1$

Differentiating with respect to $x$ ,

$$2ax + by + bx\frac{dy}{dx} + 2cy\frac{dy}{dx} = 0$$

**M1**

For stationary points, $\frac{dy}{dx} = 0$ , so $2ax + by = 0$

Multiplying the original equation by $b^2$

$$ab^2x^2 + b^3xy + b^2cy^2 = b^2$$

Thus as $by = -2ax$ , $ab^2x^2 - 2ab^2x^2 + 4a^2cx^2 = b^2$ **M1**

$$a(4ac - b^2)x^2 = b^2$$

**A1**

We require two stationary points and as $abc \neq 0$ , $b \neq 0$ and as $a > 0$ ,

$$4ac - b^2 > 0$$

giving

$$b^2 < 4ac$$

as required.

(Alternatively, as $2ax = -by$ , $(-by)^2 + 2b(-by)y + 4acy^2 = 4a$, $(4ac - b^2)y^2 = 4a$ for **M1A1**)

**E1 (4)**

(ii) $ay^3 + bx^2y + cx = 1$

Differentiating with respect to $x$ ,

$$3ay^2\frac{dy}{dx} + 2bxy + bx^2\frac{dy}{dx} + c = 0$$

**M1**

For stationary points, $\frac{dy}{dx} = 0$ , so $2bxy + c = 0$

Multiplying $ay^3 + bx^2y + cx = 1$ by $8b^3x^3$ ,

$$8ab^3x^3y^3 + 8b^4x^5y + 8b^3cx^4 = 8b^3x^3$$

So substituting for $2bxy$ ,

$$-ac^3 - 4b^3x^4c + 8b^3cx^4 = 8b^3x^3$$

**M1**

which simplifies to

$$4b^3cx^4 - 8b^3x^3 - ac^3 = 0$$

***A1**

Consider the curve,

$$y = 4b^3cx^4 - 8b^3x^3 - ac^3$$

This has stationary points given by

$$\frac{dy}{dx} = 16b^3cx^3 - 24b^3x^2 = 0$$

**M1**

i.e. $8b^3x^2(2cx-3) = 0$ so, there are only two stationary points on this quartic, which are $(0, -ac^3)$, **A1** which is a point of inflection on the y axis, **E1** and

$$\left(\frac{3}{2c}, \frac{81b^3}{4c^3} - \frac{27b^3}{c^3} - ac^3\right)$$

**A1**

which is a turning point.

So for $4b^3cx^4 - 8b^3x^3 - ac^3 = 0$ to have two solutions, if $c > 0$, the turning point needs to be a minimum below the x axis and so $\frac{81b^3}{4c^3} - \frac{27b^3}{c^3} - ac^3 < 0$ **E1** and if $c < 0$, the turning point needs to be a maximum above the x axis and so $\frac{81b^3}{4c^3} - \frac{27b^3}{c^3} - ac^3 > 0$. **E1** Thus, in either case multiplication by $4c^3$ yields

$$81b^3 - 108b^3 - 4ac^6 < 0$$

which simplifies to

$$4ac^6 + 27b^3 > 0$$

as required. **E1 (10)**

(iii) These are three simultaneous equations in two unknowns so we may solve for two of them and substitute into the third. The first equation rules out $x = 0$ as the third equation would imply

$y = 0$, given that $abc \neq 0$ and thus $ay^3 + bx^2y + cx \neq 1$ as required.

If we consider $2bxy + c = 0$ and $3ay^2 + bx^2 = 0$, the second if these implies that as $b > 0$, then $a < 0$. **E1**

Multiplying the second of these by $4by^2$, $12aby^4 + 4b^2x^2y^2 = 0$ and substituting from the first of these two equations,

$$12aby^4 + c^2 = 0$$

**M1**

Thus

$$y = \pm \sqrt[4]{\frac{-c^2}{12ab}}$$

**A1**

and so

$$x = \mp \frac{c}{2b} \sqrt[4]{\frac{12ab}{-c^2}} = \mp \sqrt[4]{\frac{-3ac^2}{4b^3}}$$

**A1**

Substituting these in $ay^3 + bx^2y + cx = 1$ , having first multiplied it by $y$ ,

that is $ay^4 + bx^2y^2 + cxy = y$

gives

$$\frac{-c^2}{12b} + \frac{c^2}{4b} - \frac{c^2}{2b} = \pm \sqrt[4]{\frac{-c^2}{12ab}}$$

which simplifies to

$$-\frac{c^2}{3b} = \pm \sqrt[4]{\frac{-c^2}{12ab}}$$

**M1**

Raising this to the power four,

$$\frac{c^8}{81b^4} = \frac{-c^2}{12ab}$$

and thus

$$4ac^6 + 27b^3 = 0$$

as required. ***A1 (6)**

(Alternative: The first two equations were combined to give $4b^3cx^4 - 8b^3x^3 - ac^3 = 0$ in part (ii). **M1**

The second and third can be combined to give $4b^3x^4 + 3ac^2 = 0$ **M1**

So, $8b^3x^3 + 4ac^3 = 0$

Thus $x = -\frac{c}{b}\sqrt[3]{\frac{a}{2}}$ **A1**

and $y = \frac{1}{\sqrt[3]{4a}}$ **A1**

So, to have a solution we require

$$3a\left(\frac{1}{\sqrt[3]{4a}}\right)^2 + b\left(-\frac{c}{b}\sqrt[3]{\frac{a}{2}}\right)^2 = 0$$

which simplifies to the required result. **M1A1**)

4. (i) Suppose

$$2^k \cosh\frac{x}{2}\cosh\frac{x}{4}\cdots\cosh\frac{x}{2^k}\sinh\frac{x}{2^k} = \sinh x$$

for some integer $k$ . **E1**

Then

$$2^{k+1} \cosh\frac{x}{2}\cosh\frac{x}{4}\cdots\cosh\frac{x}{2^{k+1}}\sinh\frac{x}{2^{k+1}} = 2\sinh x\frac{\cosh\frac{x}{2^{k+1}}\sinh\frac{x}{2^{k+1}}}{\sinh\frac{x}{2^k}}$$

(which is legitimate because $x \neq 0$ and hence $\sinh\frac{x}{2^k} \neq 0$ )

$$= \sinh x \frac{2\sinh\frac{x}{2^{k+1}}\cosh\frac{x}{2^{k+1}}}{\sinh\frac{x}{2^k}} = \sinh x \frac{\sinh\frac{x}{2^k}}{\sinh\frac{x}{2^k}} = \sinh x$$

which is the desired result for $k+1$. **M1**

$$2\cosh\frac{x}{2}\sinh\frac{x}{2} = \sinh x$$

**B1**

so, the result is true for $n = 1$ .

Hence by the principle of mathematical induction,

$$\sinh x = 2^n \cosh\frac{x}{2}\cosh\frac{x}{4}\cdots\cosh\frac{x}{2^n}\sinh\frac{x}{2^n}$$

for all positive integer $n$ .

Thus

$$\frac{\sinh x}{x}\frac{\frac{x}{2^n}}{\sinh\frac{x}{2^n}} = 2^n \cosh\frac{x}{2}\cosh\frac{x}{4}\cdots\cosh\frac{x}{2^n}\sinh\frac{x}{2^n}\,\frac{x}{x}\,\frac{1}{2^n}\,\frac{1}{\sinh\frac{x}{2^n}} = \cosh\frac{x}{2}\cosh\frac{x}{4}\cdots\cosh\frac{x}{2^n}$$

as required. This working is permissible as $x \neq 0$ , and so $\sinh\frac{x}{2^k} \neq 0$ . **E1(4)**

(ii)

$$\frac{y}{\sinh y} = \frac{y}{y+\frac{y^3}{3!}+\frac{y^5}{5!}+\cdots} = \frac{1}{1+\frac{y^2}{3!}+\frac{y^4}{5!}+\cdots} \to 1$$

as $y \to 0$ . **E1**

As, from (i),

$$\frac{\sinh x}{x}\frac{\frac{x}{2^n}}{\sinh\frac{x}{2^n}} = \cosh\frac{x}{2}\cosh\frac{x}{4}\cdots\cosh\frac{x}{2^n}$$

letting $n \to \infty$ ,

and using the result shown from the use of the Maclaurin series that

$$\frac{\frac{x}{2^n}}{\sinh\frac{x}{2^n}} \to 1$$

we have

$$\frac{\sinh x}{x} = \cosh\frac{x}{2}\cosh\frac{x}{4}\cdots\cosh\frac{x}{2^n}\cdots$$

as required. **E1 (2)**

(iii) Letting $x = \ln 2$ , $\sinh x = \frac{2-\frac{1}{2}}{2} = \frac{3}{4}$ , $\cosh\frac{x}{2} = \frac{\sqrt{2}+\frac{1}{\sqrt{2}}}{2} = \frac{3}{2\sqrt{2}}$ , $\cosh\frac{x}{4} = \frac{\sqrt{\sqrt{2}}+\frac{1}{\sqrt{\sqrt{2}}}}{2} = \frac{\sqrt{2}+1}{2\sqrt{\sqrt{2}}}$

**M1**

etc.

Thus

$$\frac{\frac{3}{4}}{\ln 2} = \frac{3}{2\sqrt{2}} \times \frac{\sqrt{2}+1}{2\sqrt{\sqrt{2}}} \times \frac{\sqrt{\sqrt{2}}+1}{2\sqrt{\sqrt{\sqrt{2}}}}\cdots$$

**M1 A1**

and

$$\frac{1}{\ln 2} = \frac{4}{2\sqrt{2}} \times \frac{\sqrt{2}+1}{2\sqrt{\sqrt{2}}} \times \frac{\sqrt{\sqrt{2}}+1}{2\sqrt{\sqrt{\sqrt{2}}}}\cdots = \frac{4}{2\sqrt{2}\sqrt{\sqrt{2}}\sqrt{\sqrt{\sqrt{2}}}\cdots} \times \frac{1+\sqrt{2}}{2} \times \frac{1+\sqrt{\sqrt{2}}}{2} \times \cdots$$

**A1**

The denominator of the first fraction is

$$2 \times 2^{\frac{1}{2}} \times 2^{\frac{1}{4}} \times \cdots = 2^{1+\frac{1}{2}+\frac{1}{4}+\cdots} = 2^2 = 4$$

**E1**

So

$$\frac{1}{\ln 2} = \frac{1+\sqrt{2}}{2} \times \frac{1+\sqrt{\sqrt{2}}}{2} \times \cdots$$

as required.

(iv) Substituting $x = \frac{i\pi}{2}$ in **M1**

$$\frac{\sinh x}{x} = \cosh\frac{x}{2}\cosh\frac{x}{4}\cdots\cosh\frac{x}{2^n}\cdots$$

and using $\sinh ix = i\sin x$ , $\cosh ix = \cos x$ , **M1**

$$\frac{\sinh\frac{i\pi}{2}}{\frac{i\pi}{2}} = \cosh\frac{i\pi}{4}\cosh\frac{i\pi}{8}\cdots\cosh\frac{i\pi}{2^{n+1}} = \frac{i}{\frac{i\pi}{2}} = \cos\frac{\pi}{4}\cos\frac{\pi}{8}\cdots\cos\frac{\pi}{2^{n+1}}\cdots$$

**M1 A1 A1**

$\cos\frac{\pi}{4}=\frac{\sqrt{2}}{2}$ , $\cos\frac{\pi}{4}=2\cos^2\frac{\pi}{8}-1$ and thus $\cos\frac{\pi}{8}=\sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}}=\frac{\sqrt{2+\sqrt{2}}}{2}$

and similarly, $\cos\frac{\pi}{16}=\sqrt{\frac{1+\frac{\sqrt{2+\sqrt{2}}}{2}}{2}}=\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}$ etc. **M1 A1 M1**

So

$$\frac{2}{\pi}=\frac{\sqrt{2}}{2}\times\frac{\sqrt{2+\sqrt{2}}}{2}\times\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots$$

as required. ***A1 (9)**

(Alternatively, by induction

$$\sin x=2^n\cos\frac{x}{2}\cos\frac{x}{4}\cdots\cos\frac{x}{2^n}\sin\frac{x}{2^n}$$

**M1 A1 E1** (as for (i))

As $\frac{y}{\sin y}\to 1$ as $y\to 0$ ,

$$\frac{\sin x}{x}=\cos\frac{x}{2}\cos\frac{x}{4}\cdots\cos\frac{x}{2^n}\cdots$$

**M1A1**

and then, substituting $x=\frac{\pi}{2}$ **M1** result follows as before **A1M1A1**. )

5. (i)

$$\int_{-a}^{a} \frac{1}{1+e^x}\,dx = \int_{-a}^{a} \frac{e^{-x}}{e^{-x}+1}\,dx = [-\ln(e^{-x}+1)]_{-a}^{a} = \ln\left(\frac{e^a+1}{e^{-a}+1}\right) = \ln e^a = a$$

**M1** **A1** ***A1(3)**

Alternative 1.

$$\int_{-a}^{a} \frac{1}{1+e^x}\,dx = \int_{-a}^{a} \frac{1+e^x}{1+e^x} - \frac{e^x}{1+e^x}\,dx = [x - \ln(1+)]_{-a}^{a} = 2a - \ln\left(\frac{e^a+1}{e^{-a}+1}\right) = 2a - \ln e^a = a$$

**M1** **A1** ***A1(3)**

Alternative 2.

Substitute $u = e^x$ ,

$$\int_{-a}^{a} \frac{1}{1+e^x}\,dx = \int_{e^{-a}}^{e^a} \frac{1}{1+u}\frac{1}{u}\,du = \int_{e^{-a}}^{e^a} \frac{1}{u} - \frac{1}{1+u}\,du = [\ln u - \ln(1+u)]_{e^{-a}}^{e^a} = 2a - \ln\left(\frac{e^a+1}{e^{-a}+1}\right) =$$

**M1** **A1**

$$2a - \ln e^a = a$$

***A1(3)**

Alternative 3.

Substitute $u = 1 + e^x$ ,

$$\int_{-a}^{a} \frac{1}{1+e^x}\,dx = \int_{1+e^{-a}}^{1+e^a} \frac{1}{u}\frac{1}{u}\,du = \int_{e^{-a}}^{e^a} \frac{1}{u} - \frac{1}{1+u}\,du = [\ln u - \ln(1+u)]_{e^{-a}}^{e^a} = 2a - \ln\left(\frac{e^a+1}{e^{-a}+1}\right) =$$

**M1** **A1**

$$2a - \ln e^a = a$$

***A1(3)**

Alternative 4.

$$\int_{-a}^{a} \frac{1}{1+e^x}\,dx = \int_{0}^{a} \frac{1}{1+e^x}\,dx + \int_{-a}^{0} \frac{1}{1+e^x}\,dx = \int_{0}^{a} \frac{1}{1+e^x}\,dx + \int_{a}^{0} \frac{1}{1+e^{-x}}.-dx$$

$$= \int_{0}^{a} \frac{1}{1+e^x}\,dx + \int_{0}^{a} \frac{1}{1+e^{-x}}\,dx = \int_{0}^{a} \frac{1}{1+e^x} + \frac{1}{1+e^{-x}}\,dx = \int_0^a \frac{1+e^{-x}+1+e^{-x}}{(1+e^x)(1+e^{-x})}\,dx$$

**M1**

$$= \int_0^a \frac{2+e^{-x}+e^{-x}}{2+e^{-x}+e^{-x}}\,dx = [x]_0^a = a$$

**A1** ***A1 (3)**

(ii)

Suppose

$$\int g(x)\,dx = G(x) + c$$

Then if

$$\int_0^a g(x)\,dx = 0 \quad \forall a \geq 0$$

$$G(a) - G(0) = 0 \quad \forall a$$

so $G(a) = constant \ \ \forall a$ and hence $\frac{dG}{dx} = g(x) = 0 \ \ \forall x \geq 0$ as required.

Alternatively, by the FTC, $g(a) = 0 \ \ \forall a \geq 0$ **E1**

$$\int_{-a}^{a} \frac{1}{1+f(x)}\,dx = a \iff \int_{-a}^{0} \frac{1}{1+f(x)}\,dx + \int_{0}^{a} \frac{1}{1+f(x)}\,dx = a$$

**M1**

$$\iff \int_{a}^{0} \frac{1}{1+f(-x)} . -dx + \int_{0}^{a} \frac{1}{1+f(x)}\,dx = a$$

**M1 A1**

$$\iff \int_{0}^{a} \frac{1}{1+f(-x)} + \frac{1}{1+f(x)} - 1 \ \ dx = 0$$

**M1 A1**

so, by stated result at start of part,

$$\iff \frac{1}{1+f(-x)} + \frac{1}{1+f(x)} - 1 = 0 \quad \forall x$$

**E1 E1**

$$\iff 1 + f(x) + 1 + f(-x) - \big(1 + f(-x)\big)\big(1 + f(x)\big) = 0$$

$$\iff f(x)\,f(-x) = 1$$

***B1 (9)**

(iii)

$$\int_{-a}^{a} \frac{h(x)}{1+f(x)}\,dx = \int_{-a}^{0} \frac{h(x)}{1+f(x)}\,dx + \int_{0}^{a} \frac{h(x)}{1+f(x)}\,dx = \int_{a}^{0} \frac{h(-x)}{1+f(-x)} . -dx + \int_{0}^{a} \frac{h(x)}{1+f(x)}\,dx$$

**M1**

$$= \int_0^a \frac{h(x)}{1+f(-x)} + \frac{h(x)}{1+f(x)}\, dx = \int_0^a h(x)\, dx$$

by the result of (ii). **M1 *A1 (3)**

(iv)

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^{-x}\cos x}{\cosh x}\, dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^{-x}\cos x}{\frac{e^x + e^{-x}}{2}}\, dx = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{1+e^{2x}}\, dx$$

**M1 A1**

$\cos x$ satisfies the conditions for $h(x)$ in part (iii) and $e^{2x}$ satisfies the conditions for $f(x)$ in part (ii). **E1**

Therefore,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^{-x}\cos x}{\cosh x}\, dx = 2\int_0^{\frac{\pi}{2}} \cos x\, dx = 2\,[\sin x]_0^{\frac{\pi}{2}} = 2$$

**M1** **A1 (5)**

6. (i)

$$\cos(\theta+\alpha)-\cos\theta=\cos\theta\cos\alpha-\sin\theta\sin\alpha-\cos\theta \approx \cos\theta\left(1-\frac{\alpha^2}{2}\right)-\sin\theta\,\alpha-\cos\theta$$

$=-\alpha\sin\theta-\frac{\alpha^2}{2}\cos\theta$ as required. **M1 *A1**

If $\sin\theta\neq 0$

$$\lim_{\alpha\to 0}\frac{\sin(\theta+\alpha)-\sin\theta}{\cos(\theta+\alpha)-\cos\theta}=\lim_{\alpha\to 0}\frac{\alpha\cos\theta-\frac{\alpha^2}{2}\sin\theta}{-\alpha\sin\theta-\frac{\alpha^2}{2}\cos\theta}=\lim_{\alpha\to 0}\frac{\cos\theta-\frac{\alpha}{2}\sin\theta}{-\sin\theta-\frac{\alpha}{2}\cos\theta}=-\cot\theta$$

**M1 A1** **A1**

(Alternative by l'Hopital, $\lim_{\alpha\to 0}\frac{\sin(\theta+\alpha)-\sin\theta}{\cos(\theta+\alpha)-\cos\theta}=\lim_{\alpha\to 0}\frac{\cos(\theta+\alpha)}{-\sin(\theta+\alpha)}=\lim_{\alpha\to 0}-\cot(\theta+\alpha)=-\cot\theta$

**M1 A1** **A1**)

If $\sin\theta=0$

$$\lim_{\alpha\to 0}\frac{\sin(\theta+\alpha)-\sin\theta}{\cos(\theta+\alpha)-\cos\theta}=\lim_{\alpha\to 0}\frac{\cos\theta}{-\frac{\alpha}{2}\cos\theta}=\lim_{\alpha\to 0}\frac{-2}{\alpha}$$

**M1**

$\to-\infty$ as $\alpha\to+0$ and $\to\infty$ as $\alpha\to-0$ **A1 (7)**

(ii) (a) If $Q_0$ is the initial point of contact of $C_1$ and $C_2$, and if $X$ is the point on $C_2$ which was initially at $Q_0$, then if $QOQ_0=\theta$, arc $QQ_0$ on $C_1$ is of length $(n-1)a\theta$ **E1** and this will equal the arc length $QX$ on $C_2$. So if $T$ is the centre of $C_2$, $QTX=(n-1)\theta$, and $TP$ makes an angle $\theta+(n-1)\theta=n\theta$ with the $x$ axis. **E1**

Thus the $x$-coordinate of $P$ is $x(\theta)=na\cos\theta+a\cos(n\theta)=a(n\cos\theta+\cos n\theta)$ as required.

Similarly, $y(\theta)=a(n\sin\theta+\sin n\theta)$. **M1 *A1 (4)**

(b) $OP=(n-1)a$ if and only if $(n\cos\theta+\cos n\theta)^2+(n\sin\theta+\sin n\theta)^2=(n-1)^2$

That is if $n^2+2n\cos(n-1)\theta+1=n^2-2n+1$ which is $\cos(n-1)\theta=-1$

**M1**

so, when $(n-1)\theta$ is an odd multiple of $\pi$ **M1**

Therefore $\theta=\frac{2r+1}{n-1}\pi$ for $r=0,1,\cdots$ **A1 (3)**

(Alternatively, $OP=(n-1)a$ only if $n\cos\theta+\cos n\theta=(n-1)\cos\theta$ i.e. $\cos n\theta=-\cos\theta$, and $n\sin\theta+\sin n\theta=(n-1)\sin\theta$ i.e. $\sin n\theta=-\sin\theta$ **M1**

Thus $\cos(n-1)\theta=-\cos\theta\cos\theta+-\sin\theta\sin\theta=-1$ so $(n-1)\theta$ is an odd multiple of $\pi$ **M1**

Result as before **A1**)

(c)

$$\lim_{\alpha\to 0}\frac{y(\theta_0+\alpha)-y(\theta_0)}{x(\theta_0+\alpha)-x(\theta_0)}=\lim_{\alpha\to 0}\frac{a(n\sin(\theta_0+\alpha)+\sin n(\theta_0+\alpha))-a(n\sin\theta_0+\sin n\theta_0)}{a(n\cos(\theta_0+\alpha)\ +\cos n(\theta_0+\alpha))-a(n\cos\theta_0\ +\cos n\theta_0)}$$

**M1**

$$=\lim_{\alpha\to 0}\frac{n\left(\alpha\ \cos\theta_0\ -\frac{\alpha^2}{2}\ \sin\theta_0\right)+\left(n\alpha\ \cos n\theta_0\ -\frac{n^2\alpha^2}{2}\ \sin n\theta_0\right)}{n\left(-\alpha\sin\theta_0\ -\frac{\alpha^2}{2}\ \cos\theta_0\right)+\left(-n\alpha\sin n\theta_0\ -\frac{n^2\alpha^2}{2}\ \cos n\theta_0\right)}$$

**M1 A1**

$$=\lim_{\alpha\to 0}\frac{\cos\theta_0+\cos n\theta_0-\frac{\alpha}{2}(\sin\theta_0+n\sin n\theta_0)}{-(\sin\theta_0+\sin n\theta_0)-\frac{\alpha}{2}(\cos\theta_0+n\cos n\theta_0)}$$

$$=\frac{\sin\theta_0\ +n\sin n\theta_0}{\cos\theta_0\ +n\cos n\theta_0}$$

as $\cos\theta_0+\cos n\theta_0\ =2\cos(n+1)\frac{\theta_0}{2}\ \cos(n-1)\frac{\theta_0}{2}$ and $(n-1)\frac{\theta_0}{2}=\frac{\pi}{2}$ so $\cos(n-1)\frac{\theta_0}{2}=0$

and similarly, $\sin\theta_0\ +\sin n\theta_0=2\sin(n+1)\frac{\theta_0}{2}\ \cos(n-1)\frac{\theta_0}{2}\ =0$

Further,

$$\sin\theta_0\ +n\ \sin n\theta_0=\sin\theta_0+n\big(\sin\big((n-1)+1\big)\,\theta_0\big)$$

$$=\sin\theta_0+n(\sin(n-1)\,\theta_0\ \cos\theta_0+\cos(n-1)\,\theta_0\ \sin\theta_0)$$

$$=(1-n)\sin\theta_0$$

and

$$\cos\theta_0\ +n\ \cos n\theta_0\ =\cos\theta_0+n(\cos(n-1)\theta_0\cos\theta_0\ -\sin(n-1)\,\theta_0\sin\theta_0)$$

$$=(1-n)\cos\theta_0$$

So

$$\lim_{\alpha\to 0}\frac{y(\theta_0+\alpha)-y(\theta_0)}{x(\theta_0+\alpha)-x(\theta_0)}=\frac{(1-n)\sin\theta_0}{(1-n)\cos\theta_0}=\ \tan\theta_0$$

**M1** **A1**

The LHS is the gradient of the tangent to the curve at $P$ and the RHS is the gradient of $OP$ , as required. **E1 (6)**

7. (i)

$$f(\boldsymbol{r}) = \boldsymbol{n} \times \boldsymbol{r} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} bz - cy \\ cx - az \\ ay - bx \end{pmatrix}$$

The x-component of $f\big(f(\boldsymbol{r})\big)$ is the x-component of $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} bz - cy \\ cx - az \\ ay - bx \end{pmatrix}$

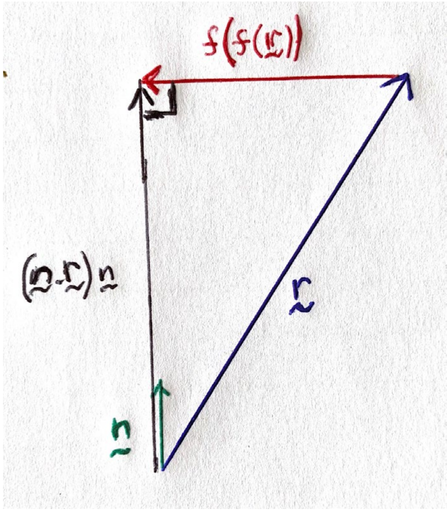
which is $b(ay - bx) - c(cx - az) = -x(b^2 + c^2) + aby + acz$ as required. **M1 *A1**

$-x(b^2 + c^2) + aby + acz = -x(a^2 + b^2 + c^2) + a^2x + aby + acz = -x + a(ax + by + cz)$

as $\boldsymbol{n}$ is a unit vector. **E1**

Similarly, the y and z -components of $f\big(f(\boldsymbol{r})\big)$ $-y + b(ax + by + cz)$ and $-z + c(ax + by + cz)$

respectively and thus $f\big(f(\boldsymbol{r})\big) = -\boldsymbol{r} + (\boldsymbol{n}.\boldsymbol{r})\boldsymbol{n}$ **M1 *A1**




**G1 G1 G1 (8)**

(ii)

$$g(\boldsymbol{n}) = \boldsymbol{n} + \sin\theta \; f(\boldsymbol{n}) + (1 - \cos\theta)\; f\big(f(\boldsymbol{n})\big)$$

$$= \boldsymbol{n} + \sin\theta \; \boldsymbol{n} \times \boldsymbol{n} + (1 - \cos\theta)\big((\boldsymbol{n}.\boldsymbol{n})\boldsymbol{n} - \boldsymbol{n}\big)$$

$$= \boldsymbol{n}$$

**M1A1**

$$g(\boldsymbol{r}) = \boldsymbol{r} + \sin\theta \; f(\boldsymbol{r}) + (1 - \cos\theta)\; f\big(f(\boldsymbol{r})\big)$$

$$= \boldsymbol{r} + \sin\theta \; \boldsymbol{n} \times \boldsymbol{r} + (1 - \cos\theta)\big((\boldsymbol{n}.\boldsymbol{r})\boldsymbol{n} - \boldsymbol{r}\big)$$

$$= \boldsymbol{r}\cos\theta + \sin\theta \; \boldsymbol{n} \times \boldsymbol{r}$$

**A1**

If $\boldsymbol{r}$ is perpendicular to $\boldsymbol{n}$, then $\boldsymbol{r}$, $\boldsymbol{n}$, and $\boldsymbol{n} \times \boldsymbol{r}$ form a mutually perpendicular vector triad.

g maps $\boldsymbol{r}$ to $\boldsymbol{r}\cos\theta + \sin\theta\ \boldsymbol{n}\ \times \boldsymbol{r}$ which represents an anticlockwise rotation by $\theta$ about an axis in the direction $\boldsymbol{n}$ as **B1** both vectors are of equal magnitude **E1** and are at angle of $\theta$ to each other **E1** and are both perpendicular to $\boldsymbol{n}$ . **E1 (7)**

(iii)

$$h(\boldsymbol{s}) = -\boldsymbol{s} - 2f(\boldsymbol{s}) = -\boldsymbol{s} - 2\big((\boldsymbol{n}.\boldsymbol{s})\boldsymbol{n} - \boldsymbol{s}\big) = \boldsymbol{s} - 2(\boldsymbol{n}.\boldsymbol{s})\boldsymbol{n}$$

So, h represents a reflection **M1** in the plane through the origin perpendicular to $\boldsymbol{n}$ **A1**

Justification. If $\boldsymbol{r}$ is as in (ii).

$$h(\boldsymbol{n}) = \boldsymbol{n} - 2(\boldsymbol{n}.\boldsymbol{n})\boldsymbol{n} = -\boldsymbol{n}$$

$$h(\boldsymbol{r}) = \boldsymbol{r} - 2(\boldsymbol{n}.\boldsymbol{r})\boldsymbol{n} = \boldsymbol{r}$$

$$h(\boldsymbol{n}\ \times \boldsymbol{r}) = \boldsymbol{n}\ \times \boldsymbol{r} - 2(\boldsymbol{n}.\boldsymbol{n}\ \times \boldsymbol{r})\boldsymbol{n} = \boldsymbol{n}\ \times \boldsymbol{r}$$

**B1**

So any vector in the plane through the origin perpendicular to $\boldsymbol{n}$ is invariant under h, **E1** and any vector in the direction of $\boldsymbol{n}$ is reversed. **E1 (5)**

8. (i)

By de Moivre,

$$\cos(k\theta) + i\ \sin(k\theta) = (\cos\theta + i\ \sin\theta)^k$$

$$= \left[\cos^k\theta - \binom{k}{2}\cos^{k-2}\theta\ \sin^2\theta + \binom{k}{4}\cos^{k-4}\theta\ \sin^4\theta - \cdots\right]$$
$$+ i\left[\binom{k}{1}\cos^{k-1}\theta\ \sin\theta - \binom{k}{3}\cos^{k-3}\theta\ \sin^3\theta + \binom{k}{5}\cos^{k-5}\theta\ \sin^5\theta - \cdots\right]$$

**M1 A1 A1**

Equating imaginary parts,

$$\sin(k\theta) = \binom{k}{1}\cos^{k-1}\theta\ \sin\theta - \binom{k}{3}\cos^{k-3}\theta\ \sin^3\theta + \binom{k}{5}\cos^{k-5}\theta\ \sin^5\theta - \cdots$$

$$= \sin\theta\ \cos^{k-1}\theta\ \left(k - \binom{k}{3}\tan^2\theta + \binom{k}{5}\tan^4\theta - \cdots\right)$$

**M1**

$$= \sin\theta\ \cos^{k-1}\theta\ \left(k - \binom{k}{3}(\sec^2\theta - 1) + \binom{k}{5}(\sec^2\theta - 1)^2 - \cdots\right)$$

as required. ***A1**

Similarly, equating real parts,

$$\cos(k\theta) = \cos^k\theta - \binom{k}{2}\cos^{k-2}\theta\ \sin^2\theta + \binom{k}{4}\cos^{k-4}\theta\ \sin^4\theta - \cdots$$

$$= \cos^k\theta\ \left(1 - \binom{k}{2}(\sec^2\theta - 1) + \binom{k}{4}(\sec^2\theta - 1)^2 - \cdots\right)$$

**B1 (6)**

(ii)

$$\sin(k\theta) = 0\ \Rightarrow \sin\theta\ \cos^{k-1}\theta\ \left(k - \binom{k}{3}(\sec^2\theta - 1) + \binom{k}{5}(\sec^2\theta - 1)^2 - \cdots\right) = 0$$

Thus, if $k$ were odd,

$$\sin\theta\ \frac{1}{a^{k-1}}\left(k - \binom{k}{3}(a^2 - 1) + \binom{k}{5}(a^2 - 1)^2 - \cdots + (-1)^{\frac{k-1}{2}}(a^2 - 1)^{\frac{k-1}{2}}\right) = 0$$

**M1**

and we are given that $\sin\theta \neq 0$

As $a$ is odd, $(a^2 - 1)$ is even. Thus

$$\left(k - \binom{k}{3}(a^2 - 1) + \binom{k}{5}(a^2 - 1)^2 - \cdots + (-1)^{\frac{k-1}{2}}(a^2 - 1)^{\frac{k-1}{2}}\right)$$

is the sum of one odd number (the first) and the remainder even, and hence is odd. **A1**

We are given that $\sin\theta \neq 0$ and because $a$ is odd, $\frac{1}{a^{k-1}} \neq 0$ , and the bracketed expression is odd and thus not zero. Hence, we have a contradiction and thus $k$ cannot be odd, and must therefore be even, as required. **E1**

If $\sin(k\theta) = 0$ , and $k$ is even, as $\sin(k\theta) = 2\sin\frac{k\theta}{2}\cos\frac{k\theta}{2}$ where $\frac{k}{2}$ is an integer, we know $\sin\frac{k\theta}{2} \neq 0$ so it would have to be that $\cos\frac{k\theta}{2} = 0$ . ***B1 (4)**

Let $\frac{k}{2} = n$ .

By the second result of (i), $\cos(n\theta) = \cos^n\theta \left(1 - \binom{n}{2}(\sec^2\theta - 1) + \binom{n}{4}(\sec^2\theta - 1)^2 - \cdots\right)$

$$= \frac{1}{a^n}\left(1 - \binom{n}{2}(a^2 - 1) + \binom{n}{4}(a^2 - 1)^2 - \cdots\right)$$

**M1**

As before, the bracketed expression is odd, being the sum of one odd number (the first which is 1) and the remainder even, and thus not zero, so $\cos(n\theta) \neq 0$ which is a contradiction. **A1**

Thus, there is no least integer $k$ for which $\sin(k\theta) = 0$ , **dM1** and hence that $k\theta = 180p$ , i.e. that $\theta = \frac{180p}{k}$ . Hence $\theta$ is irrational. **E1 (4)**

(iii) Suppose there is a positive odd integer $k$ such that $\sin(k\varphi) = 0$ and $\sin(m\varphi) \neq 0$ for all integers $m$ with $0 < m < k$.

Then $\sin(k\varphi) = \sin\varphi\,\cos^{k-1}\varphi\left(k - \binom{k}{3}\tan^2\varphi + \binom{k}{5}\tan^4\varphi - \cdots\right)$

$$= \sin\varphi\,\cos^{k-1}\varphi\left(k - \binom{k}{3}b^2 + \binom{k}{5}b^4 - \cdots\right)$$

**M1**

As before in (ii), the bracketed expression is odd and thus not zero, $\sin\varphi \neq 0$ and as

$\cot\varphi = \frac{1}{b} \neq 0$ , $\cos\varphi \neq 0$ . Hence a contradiction. **E1**

So, it would be necessary to have $k$ even.

If $\sin(k\varphi) = 0$ , and $k$ is even, as $\sin(k\varphi) = 2\sin\frac{k\varphi}{2}\cos\frac{k\varphi}{2}$ where $\frac{k}{2}$ is an integer, we know $\sin\frac{k\varphi}{2} \neq 0$ so it would have to be that $\cos\frac{k\varphi}{2} = 0$ . **E1** Let $\frac{k}{2} = n$ .

$$\cos(n\varphi) = \cos^n\varphi\left(1 - \binom{n}{2}b^2 + \binom{n}{4}b^4 - \cdots\right)$$

Once again, the bracketed expression is odd and thus not zero and $\cos\varphi \neq 0$ so we have a contradiction. **E1**

Once again, there is no value $k$ for which $\sin(k\varphi) = 0$ , **M1** i.e, that $\varphi = \frac{180p}{k}$ so $\varphi$ is irrational. **E1 (6)**

9.

Conservation of linear momentum for the collision between $A$ and $B$ gives

$$mv_1 + kmv_2 = mu$$

**M1**

i.e.

$$v_1 + kv_2 = u \qquad (1)$$

Newton's experimental law of impact gives

$$v_2 - v_1 = eu \qquad (2)$$

**M1**

$(1) - k(2)$ gives $v_1(1+k) = u(1-ke)$ and hence $v_1 = \frac{u(1-ke)}{(1+k)}$ as required. ***A1**

$(1) + (2)$ gives $v_2(k+1) = u(1+e)$ and hence $v_2 = \frac{u(1+e)}{(1+k)}$ as required. ***A1 (4)**

Time for $B$ to reach wall is $\frac{D}{\beta u}$ and the time to then return to point $\frac{1}{2}D$ from wall is $\frac{\frac{1}{2}D}{e\beta u}$

Time for $A$ to reach point $\frac{1}{2}D$ from wall is $\frac{\frac{1}{2}D}{\alpha u}$

Thus

$$\frac{\frac{1}{2}D}{\alpha u} = \frac{D}{\beta u} + \frac{\frac{1}{2}D}{e\beta u}$$

**M1 A1**

which simplifies to

$$\frac{1}{2\alpha} = \frac{1}{\beta} + \frac{1}{2e\beta} = \frac{1}{\beta}\left(1 + \frac{1}{2e}\right)$$

Hence

$$\alpha = \beta\left(\frac{e}{1+2e}\right)$$

**A1**

Thus

$$(1-ke) = (1+e)\left(\frac{e}{1+2e}\right)$$

$$ke = 1 - (1+e)\left(\frac{e}{1+2e}\right) = \frac{1+2e-e-e^2}{1+2e}$$

**M1**

and so

$$k = \frac{1 + e - e^2}{e(1 + 2e)}$$

as required. ***A1 (5)**

(ii) The first collision (between *A* and *B*) is as in part (i).

The second collision (between *B* and *C*) is as in part (i) as the ratio of masses is the same but $u$ is replaced by $\beta u$ .

Thus, after two collisions, *A* has speed $\alpha u$ , *B* has speed $\alpha\beta u$ , and *C* has speed $\beta^2 u$ . **M1 A1**

The condition that *B* and *C* collide half the distance from the wall is as in (i) $(D = 3d)$

So

$$k = \frac{1 + e - e^2}{e(1 + 2e)}$$

**E1**

Equating the times of A and B to reach the point of simultaneous collision, we have

$$\frac{\frac{5}{2}d}{\alpha u} = \frac{d}{\beta u} + \frac{\frac{3}{2}d}{\alpha\beta u}$$

**M1 A1**

Therefore

$$\frac{5}{\alpha} = \frac{2}{\beta} + \frac{3}{\alpha\beta}$$

$$5\beta = 2\alpha + 3$$

**A1**

So, substituting for $\alpha$ and $\beta$ ,

$$\frac{5(1 + e)}{(1 + k)} = \frac{2(1 - ke)}{(1 + k)} + 3$$

Thus,

$$5 + 5e = 2 - 2ke + 3 + 3k$$

$$5e = k(3 - 2e)$$

and so

$$k = \frac{5e}{3 - 2e}$$

**A1**

Equating these two expressions for $k$

$$\frac{1+e-e^2}{e(1+2e)} = \frac{5e}{3-2e}$$

**M1**

$$(3-2e)(1+e-e^2) = 5e^2(1+2e)$$

$$2e^3 - 5e^2 + e + 3 = 10e^3 + 5e^2$$

$$8e^3 + 10e^2 - e - 3 = 0$$

**A1**

Factorising we have,

$$(2e-1)(4e^2+7e+3) = 0$$

further

$$(2e-1)(e+1)(4e+3) = 0$$

**M1**

$e > 0$ so $e = \frac{1}{2}$ as required. ***A1 (11)**

10. (i)

$$BP = 2a\cos\theta$$

Thus, the extension of $BP$ is $2a\cos\theta - a = a(2\cos\theta - 1)$

**M1**

and the tension in $BP$ is $s_1 W \frac{a(2\cos\theta - 1)}{a} = s_1 W(2\cos\theta - 1)$

Resolving in the direction $BP$ , $W\sin\theta = s_1 W(2\cos\theta - 1)$

**M1 A1**

(Alternative

Resolving vertically $T_{BP}\sin\theta + T_{CP}\cos\theta = W$

Resolving horizontally $T_{BP}\cos\theta = T_{CP}\sin\theta$

Solving simultaneously $T_{BP} = W\sin\theta$

So $W\sin\theta = s_1 W(2\cos\theta - 1)$

**M1 A1** )

and hence

$$s_1 = \frac{\sin\theta}{(2\cos\theta - 1)}$$

as required. ***A1**

By symmetry,

$$s_2 = \frac{\cos\theta}{(2\sin\theta - 1)}$$

**B1 (5)**

[Both divisions are valid as both extensions are positive and so $\cos\theta > \frac{1}{2}$ and $\sin\theta > \frac{1}{2}$] @

(ii)

The GPE of the particle is $-W \times BP\sin\theta = -2Wa\sin\theta\cos\theta$

**M1 A1**

The EPE of $BP$ is

$$\frac{s_1 W\big(a(2\cos\theta - 1)\big)^2}{2a}$$

**M1**

and the EPE of *CP* is

$$\frac{s_2 W\left(a(2\sin\theta - 1)\right)^2}{2a}$$

Thus, the total potential energy of the system is

$$\frac{-Wa}{2}\ (4\sin\theta\cos\theta - s_1(2\cos\theta - 1)^2 - s_2(2\sin\theta - 1)^2$$

**A1**

$$= \frac{-Wa}{2}\left(4\sin\theta\cos\theta - \frac{\sin\theta}{(2\cos\theta - 1)}(2\cos\theta - 1)^2 - \frac{\cos\theta}{(2\sin\theta - 1)}(2\sin\theta - 1)^2\right)$$

$$= \frac{-Wa}{2}(\sin\theta + \cos\theta)$$

So

$$p = \frac{1}{2}(\sin\theta + \cos\theta)$$

**A1 (5)**

$$(\sin\theta + \cos\theta) = \sqrt{2}\ \cos(\theta - 45^o)$$

**M1 A1**

As $\cos\theta > \frac{1}{2}$ and $\sin\theta > \frac{1}{2}$, $30^o < \theta < 60^o$

The expression is a maximum when $\theta = 45^o$ when $\frac{1}{2}(\sin\theta + \cos\theta) = \frac{\sqrt{2}}{2}$ ***B1** which is attainable and a minimum when $\theta = 30^o$ or $60^o$ (from @) **M1 E1** when $\frac{1}{2}(\sin\theta + \cos\theta) = \frac{1}{4}\left(1 + \sqrt{3}\right)$ **M1** ***A1 (7)** which cannot be attained.

(Alternative 1. $(\sin\theta + \cos\theta) = \sqrt{2}\sin(\theta + 45^o)$ which, similarly, is an attainable maximum when $\theta = 45^o$ and an unattainable minimum when $\theta = 30^o$ or $60^o$

Alternative 2. Instead of using harmonic form

$\frac{dp}{d\theta} = \frac{1}{2}(\cos\theta - \sin\theta) = 0$ for stationary value **M1 A1**, giving $\tan\theta = 0$, $\theta = 45^o$ and when $\frac{1}{2}(\sin\theta + \cos\theta) = \frac{\sqrt{2}}{2}$ ***B1** which is attainable and a minimum)

So $\frac{\sqrt{2}}{2} \geq p > \frac{1}{4}\left(1 + \sqrt{3}\right)$

We require to show that $0.75 > p \geq 0.65$.

$$64 < 75\ \Rightarrow \frac{4}{25} < \frac{3}{16}\ \Rightarrow \frac{2}{5} < \frac{\sqrt{3}}{4}\ \Rightarrow 0.65 < \frac{1}{4}\left(1 + \sqrt{3}\right)$$

**M1**

$$\frac{9}{16} > \frac{1}{2} = \frac{2}{4} \Rightarrow 0.75 = \frac{3}{4} > \frac{\sqrt{2}}{2}$$

**M1**

Thus, $0.75 > \frac{\sqrt{2}}{2} \geq p > \frac{1}{4}(1 + \sqrt{3}) > 0.65$ which shows that $p = 0.7$ correct to one significant figure. ***A1 (3)**

11. (i) (a) As the coin is fair, the distribution is binomial and symmetric,

so $P(X = r) = P(X = N - r) = P(X = 2n - r)$

Therefore,

$$P(X \leq n-1) = \sum_{i=0}^{n-1} P(X = i) = \sum_{i=0}^{n-1} P(X = 2n - i) = \sum_{i=n+1}^{2n} P(X = i) = P(X \geq n+1)$$

**E1**

$$1 = P(X \leq n-1) + P(X = n) + P(X \geq n+1) = 2P(X \leq n-1) + P(X = n)$$

Hence,

$$P(X \leq n-1) = \frac{1}{2}\left(1 - P(X = n)\right)$$

**E1 (2)**

(b)

$\mu = Np = 2n \times \frac{1}{2} = n$ (or by symmetry) **B1**

$$\delta = E(|X - \mu|) = \sum_{r=0}^{n-1} (n-r)\binom{2n}{r}\left(\frac{1}{2}\right)^{2n} + \sum_{r=n+1}^{2n} (r-n)\binom{2n}{r}\left(\frac{1}{2}\right)^{2n}$$

**M1**

$$= \sum_{r=0}^{n-1} (n-r)\binom{2n}{r}\left(\frac{1}{2}\right)^{2n} + \sum_{r=n+1}^{2n} (r-n)\binom{2n}{2n-r}\left(\frac{1}{2}\right)^{2n}$$

$$= \sum_{r=0}^{n-1} (n-r)\binom{2n}{r}\left(\frac{1}{2}\right)^{2n} + \sum_{s=0}^{n-1} (n-s)\binom{2n}{s}\left(\frac{1}{2}\right)^{2n}$$

**M1**

$$= 2\sum_{r=0}^{n-1} (n-r)\binom{2n}{r}\left(\frac{1}{2}\right)^{2n} = \sum_{r=0}^{n-1} (n-r)\binom{2n}{r}\frac{1}{2^{2n-1}}$$

as required. ***A1 (4)**

(c)

$$r\binom{2n}{r} = r\frac{(2n!)}{r!\,(2n-r)!} = \frac{2n \times (2n-1)!}{(r-1)!\left((2n-1)-(r-1)\right)!} = 2n\binom{2n-1}{r-1}$$

**M1** ***A1(2)**

$$\delta = \sum_{r=0}^{n-1} (n-r)\binom{2n}{r}\frac{1}{2^{2n-1}} = \sum_{r=0}^{n-1} n\binom{2n}{r}\frac{1}{2^{2n-1}} - \sum_{r=0}^{n-1} r\binom{2n}{r}\frac{1}{2^{2n-1}}$$

**M1**

$$= \frac{1}{2^{2n-1}}\left(n\sum_{r=0}^{n-1}\binom{2n}{r} - \sum_{r=1}^{n-1} r\binom{2n}{r}\frac{1}{2^{2n-1}}\right)$$

$$= \frac{1}{2^{2n-1}}\left(n\frac{1}{2}\left(2^{2n} - \binom{2n}{n}\right) - \sum_{r=1}^{n-1} 2n\binom{2n-1}{r-1}\frac{1}{2^{2n-1}}\right)$$

**M1** **M1**

$$= \frac{n}{2^{2n-1}}\left(2^{2n-1} - \frac{1}{2}\binom{2n}{n} - 2\sum_{r=0}^{n-2}\binom{2n-1}{r}\right)$$

$$= \frac{n}{2^{2n-1}}\left(2^{2n-1} - \frac{1}{2}\binom{2n}{n} - \sum_{r=0}^{n-2}\binom{2n-1}{r} - \sum_{r=n+1}^{2n-1}\binom{2n-1}{r}\right)$$

**M1**

$$= \frac{n}{2^{2n-1}}\left\{2^{2n-1} - \frac{1}{2}\binom{2n}{n} - \left(2^{2n-1} - \binom{2n-1}{n-1} - \binom{2n-1}{n}\right)\right\}$$

$$= \frac{n}{2^{2n-1}}\left\{-\frac{1}{2}\binom{2n}{n} + 2\binom{2n-1}{n}\right\}$$

**M1**

But

$$\binom{2n-1}{n} = \frac{(2n-1)!}{n!\,(n-1)!} = \frac{2n}{2n}\frac{(2n-1)!}{n!\,(n-1)!} = \frac{1}{2}\frac{(2n)!}{n!\,n!} = \frac{1}{2}\binom{2n}{n}$$

**M1**

Thus

$$\delta = \frac{n}{2^{2n-1}}\frac{1}{2}\binom{2n}{n} = \frac{n}{2^{2n}}\binom{2n}{n}$$

as required. ***A1 (7)**

(Alternative

$$\delta = \sum_{r=0}^{n}(n-r)\binom{2n}{r}\frac{1}{2^{2n-1}} = n\sum_{r=0}^{n}\binom{2n}{r}\frac{1}{2^{2n-1}} - 2n\sum_{r=1}^{n}\binom{2n-1}{r-1}\frac{1}{2^{2n-1}}$$

**M1 A1**

$$= 2n\sum_{r=0}^{n}\binom{2n}{r}\frac{1}{2^{2n}} - 2n\sum_{s=0}^{n-1}\binom{2n-1}{s}\frac{1}{2^{2n-1}}$$

**M1**

$$= 2nP(X \le n) - 2nP(Y \le n-1)$$

(where Y is a binomial variable $\left(2n-1, \frac{1}{2}\right)$ ) **M1**

$$= n\big(1 + P(X = n)\big) - 2n \times \frac{1}{2} = n\binom{2n}{n}\frac{1}{2^{2n}}$$

**M1** **M1** ***A1** )

(ii) $\mu = Np = (2n+1) \times \frac{1}{2} = \frac{2n+1}{2}$ (or by symmetry)

$$\delta = E(|X-\mu|) = \sum_{r=0}^{2n+1}\left|r - \frac{2n+1}{2}\right|\binom{2n+1}{r}\left(\frac{1}{2}\right)^{2n+1}$$

$$= \frac{1}{2^{2n}}\sum_{r=0}^{n}\left(\frac{2n+1}{2} - r\right)\binom{2n+1}{r}$$

**M1 A1**

$$= \frac{1}{2^{2n}}\left[\frac{2n+1}{2}\sum_{r=0}^{n}\binom{2n+1}{r} - \sum_{r=0}^{n} r\binom{2n+1}{r}\right]$$

$$= \frac{1}{2^{2n}}\left[\frac{2n+1}{2} \times 2^{2n} - \sum_{r=1}^{n} r\binom{2n+1}{r}\right]$$

$$= \frac{1}{2^{2n}}\left[\frac{2n+1}{2} \times 2^{2n} - \sum_{r=1}^{n}(2n+1)\binom{2n}{r-1}\right]$$

using the first result of (i) c) **M1**

$$= \frac{1}{2^{2n}}\left[\frac{2n+1}{2} \times 2^{2n} - (2n+1)\sum_{r=0}^{n-1}\binom{2n}{r}\right]$$

$$= \frac{(2n+1)}{2^{2n}}\left[\frac{2^{2n}}{2} - \left(\frac{2^{2n} - \binom{2n}{n}}{2}\right)\right]$$

**M1**

$$= \frac{(2n+1)}{2^{2n+1}}\binom{2n}{n}$$

**A1 (5)**

which can alternatively be written as

$$= \frac{(2n+1)}{2^{2n+1}}\frac{(2n)!}{n!\,n!} = \frac{(2n+1)!}{2^{2n+1}n!\,n!} = \frac{(2n+1)!\,(n+1)}{2^{2n+1}(n+1)!\,n!} = \frac{(n+1)}{2^{2n+1}}\binom{2n+1}{n}$$

(Alternative

$$\delta = \sum_{r=0}^{n} \left( \frac{2n+1}{2} - r \right) \binom{2n+1}{r} \frac{1}{2^{2n}}$$

**M1 A1**

$$= (2n+1) \sum_{r=0}^{n} \binom{2n+1}{r} \frac{1}{2^{2n+1}} - (2n+1) \sum_{r=1}^{n} \binom{2n}{r-1} \frac{1}{2^{2n}}$$

**M1**

$$= (2n+1) \frac{1}{2} - (2n+1) \sum_{s=0}^{n-1} \binom{2n}{s} \frac{1}{2^{2n}}$$

$$= (2n+1) \frac{1}{2} - (2n+1) \left( \frac{1}{2} - \frac{1}{2} \binom{2n}{n} \frac{1}{2^{2n}} \right)$$

**M1**

$$= \frac{(2n+1)}{2^{2n+1}} \binom{2n}{n}$$

**A1** )

12. (i)

If $AOB = \theta$ , then the probability distribution function for $\theta$ $\quad f(\theta) = \frac{1}{2\pi}$ on $[0,2\pi]$

$$AB = 2a\sin\frac{\theta}{2}$$

**M1 A1**

$$E(AB) = \int_0^{2\pi} 2a\sin\frac{\theta}{2}\,\frac{1}{2\pi}\;d\theta$$

**M1 A1**

$$= \frac{2a}{\pi}\left[-\cos\frac{\theta}{2}\right]_0^{2\pi}$$

$$= \frac{4a}{\pi}$$

**A1 (5)**

(Alternatives replace $\theta$ with $2\varphi$ , $f(\varphi) = \frac{1}{\pi}$ on $[0,\pi]$ ,

or minor segment $AOB = 2\varphi$ , $f(\varphi) = \frac{2}{\pi}$ on $\left[0,\frac{\pi}{2}\right]$ )

(ii)

$$P(R \leq x) = \frac{\pi x^2}{\pi a^2} = \frac{x^2}{a^2}$$

**M1**

Therefore

$$f_R(x) = \frac{2x}{a^2}$$

for $0 \leq x \leq a$ **A1 (2)**

If the ends of the chord are $X$ and $Y$ , then $OXY$ is an isosceles triangle so

$$XY = 2\sqrt{a^2 - R^2\sin^2 t}$$

**M1 A1 (2)**

$$L(t) = \int_0^a 2\sqrt{a^2 - x^2\sin^2 t}\;\frac{2x}{a^2}\;dx$$

**M1 A1**

$$= \left[-\frac{4}{3a^2\sin^2 t}\left(a^2 - x^2\sin^2 t\right)^{\frac{3}{2}}\right]_0^a$$

$$= -\frac{4}{3a^2\sin^2 t}\left(a^3\cos^3 t - a^3\right)$$

$$= \frac{4a(1-\cos^3 t)}{3\sin^2 t}$$

as required. **dM1 *A1 (4)**

$$\frac{(1-\cos^3 t)}{\sin^2 t} = \frac{(1-\cos^3 t)}{(1-\cos^2 t)} = \frac{1+\cos t+\cos^2 t}{1+\cos t} = \frac{1}{1+\cos t} + \frac{\cos t\,(1+\cos t)}{1+\cos t}$$

**M1**

$$= \frac{1}{2\cos^2\frac{t}{2}} + \cos t = \cos t + \frac{1}{2}\sec^2\frac{t}{2}$$

**M1 *A1 (3)**

(Alternative

$$\frac{(1-\cos^3 t)}{\sin^2 t} = \frac{1-\cos t+\cos t\,(1-\cos^2 t)}{\sin^2 t} = \frac{2\sin^2\frac{t}{2}}{4\sin^2\frac{t}{2}\cos^2\frac{t}{2}} + \cos t = \frac{1}{2}\sec^2\frac{t}{2} + \cos t$$

**M1** **M1** ***A1** )

giving

$$L(t) = \frac{4a}{3}\left(\cos t + \frac{1}{2}\sec^2\frac{t}{2}\right)$$

(iii)

$$E\big(L(T)\big) = \int_0^{\frac{\pi}{2}} \frac{4a}{3}\left(\cos t + \frac{1}{2}\sec^2\frac{t}{2}\right)\frac{2}{\pi}\,dt$$

**M1 A1**

$$= \frac{8a}{3\pi}\left[\sin t + \tan\left(\frac{t}{2}\right)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{16a}{3\pi}$$

**M1 A1 (4)**